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Pochhammer-Chree Waves: Spectral Analysis of Axially Symmetric Modes

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Abstract

The exact solutions of the linear Pochhammer-Chree equation for propagating harmonic waves in a cylindrical rod, are analyzed. Limiting velocities of the Pochhammer-Chree waves are discussed. Spectral analysis of the matrix dispersion equation for longitudinal axially symmetric modes of the Pochhammer-Chree waves is performed. Analytical expressions for displacement fields are derived.

Keywords: Pochhammer-chree waves; Polarization; Dispersion; Spectral analysis

Introduction

The equation for propagating harmonic waves in a cylindrical rod, now known as the Pochhammer-Chree equation, was for the first time derived [1-3]. However, the corresponding solutions binding the phase or group speed with frequency remained unexplored until mid of the last century, when the first branches of the dispersion curves were obtained numerically [4-22]. In references [4-20] longitudinal axially symmetric modes were explored, and in [21,22] flexural and torsional modes were also considered. According to Meeker et al. [16] the axially symmetric longitudinal modes are denoted by *L* (*0, m*), where *m* is the mode number.

In [4-6] by asymptotic methods were obtained analytical formulas for both short-wave ($c_{1,\ lim}$) and long-wave ($c_{2,\ lim}$) limits for the phase speed for the lowest (fundamental) branch of the longitudinal axially symmetric modes. Following [6,15], the short-wave limit speed ($c_{1,\ lim}$) at ω→∞

$$c_{1,\ lim} = c_R \tag{1}$$

This coincides with Rayleigh wave speed (c_R), while the long-wave limit speed ($c_{2,\ lim}$) yields [15]

$$c_{2,lim} = \sqrt{\frac{E}{\rho}} \tag{2}$$

where *E* is Young's modulus, and ρ is the material density. In [6,15] the long-wave limit $c_{2,\ lim}$ was named as the "rod" wave speed.

Dispersion curves related to higher axially symmetric modes were studied in [4-20]. In the velocity of longitudinal waves in cylindrical bars [8] the first several roots of the dispersion equation were (numerically) obtained and it was revealed that some of the roots were complex relating to attenuating modes. Beside dispersion curves, variation of the displacement magnitudes along radius of the rod for the first three *L* (*0, m*) modes at fixed Poisson's ratio ν =0.3317 was analyzed in an experimental and theoretical investigation of elastic wave propagation in a cylinder [19].

One of the interesting peculiarities of propagating *L* (*0, m*), *m* >1 modes at γ → 0, where γ is the wave number (γ=2π/λ, λ is the wavelength), corresponds to the zero slope of the dimensionless frequency Ω [15]:

$$\lim_{\gamma\to 0}\frac{\partial\Omega}{\partial\gamma} = 0 \tag{3}$$

In (3) $\Omega = \omega R/c_2$ with ω being circular frequency, *R* is radius of the rod cross section, and c_2 speed of the bulk shear wave. Actually, condition (3) means presence of the horizontal asymptote in the dispersion relation ω(*c*) at the phase speed c→∞ for higher longitudinal axially symmetric modes. Resemblance with the dispersion curves at c→∞ for higher modes of Lamb waves can be observed [23].

Extensions of the Pochhammer-Chree waves to helical waves (longitudinal axially symmetric modes) that relate to non-integer coefficients at the angle coordinate in the corresponding potentials were analyzed [24-26].

Principle Equations

Equation of motion for an isotropic medium at absence of body forces can be represented in a form

$$c_1^2\nabla\operatorname{div}\mathbf{u} - c_2^2\operatorname{rot}\operatorname{rot}\mathbf{u} = \partial_{tt}^2\mathbf{u} \tag{4}$$

where u is the displacement field, c_1, c_2 are speeds of bulk longitudinal and shear waves respectively, and

$$c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}},\quad c_2 = \sqrt{\frac{\mu}{\rho}} \tag{5}$$

In (5) λ, μ are Lame's constants, and ρ is a material density.

The Helmholtz representation for the displacement field u yields

$$\mathbf{u} = \nabla\Phi + \operatorname{rot}\mathbf{\Psi} \tag{6}$$

Where Φ and are scalar and vector potentials respectively. In cylindrical coordinates representation (6) for the physical components of the displacement field, becomes

$$\begin{aligned} u_r &= \frac{\partial\Phi}{\partial r} + \frac{1}{r}\frac{\partial\Psi_z}{\partial\theta} - \frac{\partial\Psi_\theta}{\partial z} \\ u_\theta &= \frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\partial\Psi_r}{\partial z} - \frac{\partial\Psi_z}{\partial r} \\ u_z &= \frac{\partial\Phi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\Psi_\theta) - \frac{1}{r}\frac{\partial\Psi_r}{\partial\theta} \end{aligned} \tag{7}$$

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In (7) it is assumed that coordinate z directs along central axis of the rod. It is assumed that the displacement field is axially symmetric, that yields

$$u_\theta = 0 \tag{8}$$

Substituting (6) into equation of motion (4) yields

$$c_1^2 \Delta \Phi = \ddot{\Phi}, c_2^2 \Delta \boldsymbol{\Psi} = \ddot{\boldsymbol{\Psi}} \tag{9}$$

For a harmonic wave propagating along axis z, potentials (9) can be represented in a form

$$\Phi = \Phi_0(\boldsymbol{x}')e^{i\gamma(z-ct)}, \boldsymbol{\Psi} = \boldsymbol{\Psi}_0(\boldsymbol{x}')e^{i\gamma(z-ct)} \tag{10}$$

where, as before, γ is the wave number related to the phase speed c and circular frequency ω by equation

$$\gamma = \frac{\omega}{c} \tag{11}$$

In (10) $\boldsymbol{x}'$ is the (vector) coordinate in the cross section of a rod ($\mathbf{x}' = \mathbf{x} - (\mathbf{n} \cdot \mathbf{x})\mathbf{n}$), $\mathbf{n}$ is the wave vector; and z= n. x.

Substituting representations (10) into Eqs. , yields the Helmholtz equations for the potentials:

$$\Delta\Phi_0 + \left(\frac{c^2}{c_1^2} - 1\right)\gamma^2 \Phi_0 = 0, \quad \Delta\boldsymbol{\Psi}_0 + \left(\frac{c^2}{c_2^2} - 1\right)\gamma^2 \boldsymbol{\Psi}_0 = 0 \tag{12}$$

Axial symmetry of Φ_0 ensures [13,14]

$$\frac{\partial \Phi_0}{\partial \theta} = 0 \tag{13}$$

Equations (12), (13) result in Bessel's equation:

$$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} \Phi_0(r) + \left(\frac{c^2}{c_1^2} - 1\right)\gamma^2 \Phi_0(r) = 0 \tag{14}$$

where c is the phase speed. The solution of Eq. (14) can be represented in terms of the corresponding Bessel functions

$$\Phi_0(r) = C_1 J_0(q_1 r) + C_2 Y_0(q_1 r) \tag{15}$$

where C_k, k =1, 2 are the unknown complex coefficients, and

$$q_1^2 = \left(\frac{c^2}{c_1^2} - 1\right)\gamma^2 \tag{16}$$

Axial symmetry of potential ψ_0 is satisfied by the following equations [13,14]

$$\frac{\partial \Psi_r}{\partial \theta} = \frac{\partial \Psi_\theta}{\partial \theta} = \frac{\partial \Psi_z}{\partial \theta} = 0 \tag{17}$$

Equations (12), (17) yield Bessel equations (for physical components)

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} \Psi_r(r) + \left(\left(\frac{c^2}{c_2^2} - 1\right)\gamma^2 - \frac{1}{r^2}\right)\Psi_r(r) = 0 \\
&\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} \Psi_\theta(r) + \left(\left(\frac{c^2}{c_2^2} - 1\right)\gamma^2 - \frac{1}{r^2}\right)\Psi_\theta(r) = 0 \\
&\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} \Psi_z(r) + \left(\frac{c^2}{c_2^2} - 1\right)\gamma^2 \Psi_z(r) = 0
\end{aligned} \tag{18}$$

The solutions of Eqs. (18) are

$$\begin{aligned}
\Psi_\theta(r) &= C_3 J_1(q_2 r) + C_4 Y_1(q_2 r) \\
\Psi_r(r) &= C_5 J_1(q_2 r) + C_6 Y_1(q_2 r) \\
\Psi_z(r) &= C_7 J_0(q_2 r) + C_8 Y_0(q_2 r)
\end{aligned} \tag{19}$$

In (19) C_k, k = 3 ,…,8 are the unknown complex coefficients, and

$$q_2^2 = \left(\frac{c^2}{c_2^2} - 1\right)\gamma^2 \tag{20}$$

Axial symmetry of the vector potential ψ imposes another restriction [14,16]:

$$\psi_r = \psi_z = 0 \tag{21}$$

Now, accounting (7), (8), (15), (16), (21), the desired vector field corresponding to the propagating longitudinal axially symmetric harmonic wave, becomes [19].

$$\begin{aligned}
u_r &= -\left[q_1\left(C_1 J_1(q_1 r) + C_2 Y_1(q_1 r)\right) + i\gamma\left(C_3 J_1(q_2 r) + C_4 Y_1(q_2 r)\right)\right]e^{i\gamma(z-ct)} \\
u_\theta &= 0 \\
u_z &= \left[i\gamma\left(C_1 J_0(q_1 r) + C_2 Y_0(q_1 r)\right) + q_2\left(C_3 J_0(q_2 r) + C_4 Y_0(q_2 r)\right)\right]e^{i\gamma(z-ct)}
\end{aligned} \tag{22}$$

Since components (22) vector field should be finite at r = 0 and noting that at r0 Bessel's functions Y_0, Y_1 are unbounded, the final representation flows out from (22).

$$\begin{aligned}
u_r &= -\left[q_1 C_1 J_1(q_1 r) + i\gamma C_2 J_1(q_2 r)\right]e^{i\gamma(z-ct)} \\
u_\theta &= 0 \\
u_z &= \left[i\gamma C_1 J_0(q_1 r) + q_2 C_2 J_0(q_2 r)\right]e^{i\gamma(z-ct)}
\end{aligned} \tag{23}$$

At deriving (23) from (23), the constant C_3 is denoted by C_2.

Remark 2.1: Expressions (23) that at r = 0 the natural condition u_r = 0 is satisfied since $J_1(0) = 0$. At the same time $J_0(0) = 1$, so U_Z at r = 0 takes the form

$$u_z = \left[i\gamma C_1 + q_2 C_2\right]e^{i\gamma(z-ct)} \tag{24}$$

Expression (24) gives the following necessary and sufficient condition for vanishing u_z at r = 0

$$i\gamma\, C_1 = -q_2 C_2 \tag{25}$$

Dispersion Equation

Traction free boundary conditions on a lateral cylindrical surface at $r = R$ have the form

$$\mathbf{t}_\nu \equiv \left(\lambda(\mathrm{tr}\,\boldsymbol{\varepsilon}) \quad + 2\mu\boldsymbol{\varepsilon}\cdot \quad\right)\Big|_{r=R} = 0 \tag{26}$$

where v is the (outward) surface normal.

Substituting the displacement representation (23) into boundary conditions (26), yields the following equations written up to exponential multiplier $e^{i\gamma(z-ct)}$)

$$\begin{aligned}
t_{rr} &\equiv \lambda I_\varepsilon + 2\mu\varepsilon_{rr} = -\left[\begin{array}{l} \lambda\left(q_1^2 + \gamma^2\right) J_0(q_1 r) C_1 + \\ + \dfrac{2\mu}{r}\left[\begin{array}{l} q_1 C_1\left(q_1 r J_0(q_1 r) - J_1(q_1 r)\right) + \\ + i\gamma C_2\left(q_2 r J_0(q_2 r) - J_1(q_2 r)\right) \end{array}\right] \end{array}\right]_{r=R} = 0 \\
t_{rz} &\equiv 2\mu\varepsilon_{rz} = -\mu\left[\begin{array}{l} i\gamma\left[q_1 C_1 J_1(q_1 r) + i\gamma C_2 J_1(q_2 r)\right] \\ + \left[i\gamma q_1 C_1 J_1(q_1 r) + q_2^2 C_2 J_1(q_2 r)\right] \end{array}\right]_{r=R} = 0
\end{aligned} \tag{27}$$

Equations (27) result in the desired dispersion equation

$$\det A=0 \tag{28}$$

where A is a square and generally non-symmetric 2×2 matrix with complex coefficients:

$$\begin{aligned}
A_{11} &= -\left(\left(q_1^2+\gamma^2\right)\frac{c_1^2}{c_2^2}-2\gamma^2\right)J_0(q_1R)+\frac{2q_1}{R}J_1(q_1R)\\
A_{12} &= -\frac{2i\gamma}{R}\left(q_2RJ_0(q_2R)-J_1(q_2R)\right)\\
A_{21} &= -2i\gamma q_1 J_1(q_1R)\\
A_{22} &= -\left(q_2^2-\gamma^2\right)J_1(q_2R)
\end{aligned} \tag{29}$$

At deriving (29) from (27) the following identity was used

$$\frac{\lambda}{\mu}=\frac{c_1^2}{c_2^2}-2 \tag{30}$$

Two-dimensional (right) eigenvectors related to vanishing eigenvalues (kernel eigenvectors) of matrix A define polarization of the corresponding Pochhammer-Chree waves.

Remark 5.1: A) Dispersion equation (28) can have both real and imaginary and complex roots [13-15]. Real roots correspond to propagating modes without attenuation in z direction, while imaginary and complex roots correspond to non-propagating modes attenuating in z direction. Herein, only propagating modes will be analyzed.

B) Substituting components (29) into Eq. (28) yields the dispersion equation in the form [19]

$$\begin{aligned}
&4\gamma^2 q_1q_2J_0(q_2R)J_1(q_1R)-\frac{2q_1}{R}\left(q_2^2+\gamma^2\right)J_1(q_1R)J_1(q_2R)+\\
&+\left(q_2^2-\gamma^2\right)\left((q_1^2+\gamma^2)\frac{c_1^2}{c_2^2}-2\gamma^2\right)J_0(q_1R)J_1(q_2R)=0
\end{aligned} \tag{31}$$

Spectral Analysis of Matrix A

Spectral analysis of matrix A splits into two cases.

Matrix A is (semi) simple

That is the case when matrix A contains no Jordan blocks, and hence has two distinct eigenvectors. Spectral decomposition of A yields:

$$\vec{\alpha}_1 \leftrightarrow \lambda_1,\quad \vec{\alpha}_2 \leftrightarrow \lambda_2 \tag{32}$$

Where λ_1, λ_2 are right eigenvalues of matrix A, and $\vec{\alpha}_1, \vec{\alpha}_2$ are two-dimensional eigenvectors. The characteristic equation for matrix A written in the form

$$\det(A-\lambda I)=0 \tag{33}$$

where I is the unit diagonal matrix, yields the following representations for eigenvalues

$$\lambda_{1,2}=s\pm d \tag{34}$$

where

$$s=\frac{A_{11}+A_{22}}{2},\quad d=\sqrt{f^2+A_{12}A_{21}},\quad f=\frac{A_{11}-A_{22}}{2} \tag{35}$$

Note, that coefficients A_{ij} in (35) are defined by (29).

The corresponding (normed) eigenvectors have the form

$$\vec{\alpha}_{1,2}=\begin{pmatrix}\dfrac{f\pm d}{\sqrt{|A_{21}|^2+|f\pm d|^2}}\\ \dfrac{A_{21}}{\sqrt{|A_{21}|^2+|f\pm d|^2}}\end{pmatrix} \tag{36}$$

Analyses of expressions (34) and (36) allow formulating

Proposition 6.1

a) The necessary and sufficient condition for simplicity of matrix A is

$$d\neq 0 \tag{37}$$

where discriminant d is defined by (35).

b) Condition for degeneracy of matrix A takes the form

$$A_{11}A_{22}=A_{12}A_{21} \tag{38}$$

Proof

a) Expression (34) reveals, that condition (37) gives a necessary and sufficient condition for simplicity of the considered matrix.

b) Due to (34), condition of degeneracy takes the form:

$$s^2=d^2 \tag{39}$$

Equation (39) with account of (35) yields the desired Eq.(38).

Remark 6.1: The straightforward analysis reveals that condition (38) is equivalent to the dispersion equation (28).

Matrix A is non-semisimple (contains Jordan block)

Condition for non-simplicity of matrix A following from expression (34), yields

$$d=0 \tag{40}$$

At (40) the spectral decomposition of matrix A results

$$\begin{pmatrix}\dfrac{f}{\sqrt{|A_{21}|^2+|f|^2}}\\ \dfrac{A_{21}}{\sqrt{|A_{21}|^2+|f|^2}}\end{pmatrix}\leftrightarrow \lambda_{1,2}=s \tag{41}$$

Thus, at (40) matrix A becomes not only non-simple matrix, but non-semisimple as well, since it has only one (right) eigenvector (41).

At (40) and in view of (34) the double degeneracy of A is equivalent to

$$S=0 \tag{42}$$

Now, taking into account (35), the following proposition flows out

Proposition 6.2. a) The necessary and sufficient condition for non-semisimplicity of matrix A is

$$f^2=-A_{12}A_{21} \tag{43}$$

b) At condition (43) double degeneracy of matrix A takes the form

$$A_{11}=-A_{22} \tag{44}$$

Proof

a) Proposition 6.1.a ensures that at (40) matrix A becomes non-simple. But, at (40) both eigenvectors coincide due to (36), so, actually

A becomes non-semisimple. Then, substituting expressions (35) into (40) yields condition (43).

b) Due to (34), degeneracy of matrix A at (40) is equivalent to

$$s = 10 \qquad (45)$$

But, (45) is equivalent to (44).

Remark 6.2: For the considered case of degeneracy of the non-semisimple matrix, the corresponding dispersion equation takes the form

$$\left(q_2^2-\gamma^2\right)^2 \left(J_1(q_2R)\right)^2 - \gamma^2 q_1 q_2 J_0(q_2R) J_1(q_1R) + \frac{\gamma^2 q_1}{R} J_1(q_2R) J_1(q_1R) = 0 \qquad (46)$$

Displacement Fields

Components of the kernel eigenvectors (36), (41), that correspond to vanishing eigenvalues, are coefficients C_1, C_2 in expressions (23). Depending on the spectral properties of matrix A, two cases are considered.

Matrix A is (semi) simple

Substituting components of the kernel eigenvector (36) that corresponds to vanishing eigenvalue (34) into (23) at condition (38), yields

$$\begin{aligned} u_r &= \frac{-\left[q_1(f \pm d)J_1(q_1r) + i\gamma A_{21} J_1(q_2r)\right]}{\sqrt{|A_{21}|^2 + |f \pm d|^2}} e^{i\gamma(z-ct)} \\ u_z &= \frac{\left[i\gamma(f \pm d)J_0(q_1r) + q_2 A_{21} J_0(q_2r)\right]}{\sqrt{|A_{21}|^2 + |f \pm d|^2}} e^{i\gamma(z-ct)} \end{aligned} \qquad (47)$$

where *f*, *d* are defined by (35), and coefficients A_{ij} of matrix A are defined by(29). In and further vanishing component u is not present.

Proposition 7.1

For (semi) simple matrix A the displacement component u_z vanishes at $r = 0$ and at $c = c_2$ regardless of frequency.

Proof: For the considered case condition (25) takes the form

$$i\gamma\left(f \pm d\right) = -q_2 A_{21} \qquad (48)$$

Equation (51) with account of (35) can be transformed to the equivalent equation:

$$i\gamma q_2\left(A_{11} - A_{22}\right) + q_2^2 A_{21} + \gamma^2 A_{12} = 0 \qquad (49)$$

Substituting expressions (29) into (52) at $c = c_2$ ensures vanishing u_z at $r = 0$.

Corollary: For the considered simple matrix A, expressions (47) are applicable for any axially symmetric mode

L(0, *m*), $m > 0$.

Matrix A is non-semisimple (contains Jordan block)

Substituting components of the kernel eigenvector (41) into (23) with account of conditions of degeneracy (43), (44), yield

$$\begin{aligned} u_r &= \frac{-\left[q_1 f J_1(q_1r) + i\gamma A_{21} J_1(q_2r)\right]}{\sqrt{|A_{21}|^2 + |f|^2}} e^{i\gamma(z-ct)} \\ u_z &= \frac{\left[i\gamma f J_0(q_1r) + q_2 A_{21} J_0(q_2r)\right]}{\sqrt{|A_{21}|^2 + |f|^2}} e^{i\gamma(z-ct)} \end{aligned} \qquad (50)$$

Proposition 7.2

For non-semisimple matrix A the displacement component u_z does not vanish at $r = 0$ and at $c_= c_2$ regardless of frequency.

Proof: For the considered case condition (25) takes the form

$$i\gamma f = -q_2 A_{21} \qquad (51)$$

Equation (52) with account of (35) can be transformed to the equivalent equation

$$i\gamma\left(A_{11} - A_{22}\right) + 2q_2 A_{21} = 0 \qquad (52)$$

Substituting (29) into (49) at $c_= c_2$ reveals that condition (52) does not hold.

Corollary: For the considered non-semisimple matrix A, expressions (50) are applicable for any axially symmetric mode, *L* (0, *m*), *m* 0.

Displacement amplitudes on a lateral surface

Normalized amplitudes U_r, U_z of the displacement fields on a cylinder lateral surface at $r = R$ can be defined by the following formulas

$$\begin{aligned} U_r &\equiv \left.\frac{|u_r|}{|u_r| + |u_z| + 1}\right|_{r=R} \\ U_z &\equiv \left.\frac{|u_z|}{|u_r| + |u_z| + 1}\right|_{r=R} \end{aligned} \qquad (53)$$

Remark 7.3: Generally, amplitude values of the components $|u_z|$ and $|u_z|$ can simultaneously vanish at some value of the radius, and particularly they can vanish at $r = R$. For this reason unity is added to the denominators in (53).

Conclusion

The exact solutions of the linear Pochhammer-Chree equation for propagating harmonic axisymmetric longitudinal waves L (0, *m*) in a cylindrical rod were analyzed. Limiting velocities of the Pochhammer-Chree waves are discussed.

Spectral analysis of the matrix dispersion equation for longitudinal axially symmetric modes *L* (0, *m*), $m > 0$ of Pochhammer-Chree waves was done, revealing that no longitudinal modes can propagate at c_2 phase speed.

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