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Dynamic warehouse size planning with demand forecast and contract flexibility

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This paper develops a dynamic warehouse planning model incorporating demand forecast and contract flexibility, and addresses how demand forecast and contract flexibility affect warehouse size planning. In this model, a manager announces a nominal size of the warehouse space to rent before the planning horizon begins (strategic decision), and determines the ordering quantity and actual warehouse size during the horizon (operational decision). In particular, the manager can adjust the actual warehouse size within a range according to dynamically updating demand forecast during the horizon, which reflects the contract flexibility. We start with the characterisation of the operational decision. For any given nominal size, we show the monotonicity of optimal inventory replenishment and warehousing decisions w.r.t. demand forecast and contract flexibility. However, this monotonicity does not necessarily hold for the strategic choice of the nominal size. Finally, a case study is presented to investigate the interaction between demand forecast information and contract flexibility. We find that the value of demand forecast can be enhanced as the contract flexibility improves. However, more forecasted demands do not imply higher value of contract flexibility.

Keywords: warehousing systems; inventory management; warehouse sizing; inventory replenishment; demand forecast; contract flexibility

1. Introduction

Warehouse size planning is one of the critical steps in supply chain management (Pang and Chan 2016). Because of the space demand uncertainty, warehouse sizing becomes a hard issue for managers. Poor warehouse size planning can have a significant impact on the efficiency of the operation. An excess of storage space results in a higher storage cost caused by empty warehouse space. On the other hand, lack of storage space can lead to extra cost of using overflow warehouse and longer response time. To reduce the loss caused by the space demand uncertainty, managers often integrate (i) storage demand forecast and (ii) contract flexibility into warehouse size planning. Storage demand forecast, originating from sale plans or product demand forecasts, help reduce the storage demand uncertainty. Moreover, innovative flexible contract allows managers dynamically accommodate warehouse size to storage demand variation. In this paper, we aim to study a dynamic warehouse sizing problem together with (i) and (ii), which is rarely addressed by the existing literature.

This paper is inspired by a real-life problem first studied by Choi (2009). In this problem, an international manufacturer rents warehouse space provided by Third-Party Logistics service providers (TPLs) to store her products sold in overseas markets. When planning the size of the warehouse space to rent, this manufacturer often takes sale plans and product demand forecasts into consideration. Meanwhile, she often uses a flexible contract which allows her to adjust the space as the inventory state or demand forecast updates.

Based on the above-mentioned problem, we develop a single product and periodic review model with a planning horizon of N periods. In this model, the manager makes both strategic and operational level decisions (Van Den Berg 1999; Rouwenhorst et al. 2000). On the strategic level, the manager signs a contract with a TPL announcing a nominal size of space to rent based on initial demand forecast before the planning horizon begins. On the operational level, the manager makes replenishment decisions at the beginning of each period during the horizon. Meanwhile, the manager can adjust the warehouse space within a range, which is determined by the nominal size and a pair of parameters (expansion and reduction parameters), according to the inventory state and updated demand forecast. In particular, this pair of parameters measure the contract flexibility. For example, when both of the expansion and reduction parameters are equal to zero, the contract degenerates into the traditional contract without flexibility (White and Francis 1971); when the expansion parameter is infinitely large and the reduction one is equal to 1, the contract becomes fully flexible (Lowe, Francis, and Reinhardt 1979). The objective is to minimise the total discounted inventory and warehousing costs by optimising the operational decisions across the horizon and strategic decisions before the planning horizon begins.

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By studying such a dynamic warehouse sizing problem, we contribute to providing answers for the following questions, together with insights and implications for warehouse managers.

How does the demand forecast affect the optimal replenishment and sizing decisions? For any given nominal size, we show that the optimal operational decisions during the horizon follow a state-dependent base stock policy. In particular, the optimal order quantity and warehouse size are increasing with the forecasted demand of the products. In other words, the manager should order more products and rent more space when more product demands are forecasted to arrive. This monotonicity result characterises the positive role of product demand forecast on the operational decision-making. However, this monotonicity result does *not* necessarily hold for the strategic decisions. An interesting finding is that the optimal nominal size increases with the forecasted demand of the last period, however, may decrease with the forecasted demand of the other periods.

How does the contract flexibility affect the optimal replenishment and sizing decisions? For any given nominal size, we show that the optimal order quantity and warehouse size are increasing (decreasing) with the flexibility expansion (reduction) parameter. This monotonicity result implies that the contract flexibility enhances the flexibility in operational decision-making. Unfortunately, this monotonicity does *not* necessarily hold for the strategic choice of the nominal space, either. We identify sufficient conditions under which the optimal nominal size is increasing with β, which indicates that the manager is allowed to announce a larger nominal size when she has more flexibility in reducing the storage capacity in future.

What is the interaction between the contract flexibility and demand forecast? We present a case study to investigate the value of contract flexibility (VCF) and demand forecast and their iteration. The result shows that the value of demand forecast (VDF) increases with the contract flexibility, that is, the manager can make better use of the demand forecast when she has more contract flexibility. However, the VCF does *not* necessarily increase with the forecasted demand. For example, when the forecasted demand is relatively small, the flexibility in capacity reduction is more valuable for the manager (because the demanded space is also relatively small). As the forecasted demand increases, this flexibility becomes less significant, i.e. the VCF decreases.

The remainder of this paper is organised as follows. Section 2 reviews relevant literature. Section 3 provides a formulation of the model. Section 4 introduces a useful definition called $L^\natural$-convexity. In the first part of Section 5, we characterise how the demand forecast and operational flexibility affect the operational decisions during the horizon for any given nominal size; in the second part, we analyse the effect of the demand forecast and operational flexibility on optimal choice of the nominal size. A case study is presented in Section 6 to illustrate the interaction between the demand forecast and operational flexibility. Section 7 presents the concluding remarks and some potential extensions.

2. Literature review

This paper is closely related to two streams of literature on (i) warehouse sizing problem and (ii) dynamic inventory problem with demand forecast.

Warehouse sizing problem is basically a planning problem of choosing storage capacity subject to variable storage demands and costs. The earliest research work on warehouse sizing problem dates back to Cahn (1948), and there has been a large body of literature on the warehouse sizing problem. The existing studies can be classified into two categories: static and dynamic. In the static problems, researchers often address how much warehouse space to construct or lease by a long-term contract (Rosenblatt and Roll 1984; Cormier and Gunn 1996a, 1996b; Mark, Jihong, and Chung-Piaw 2001; Heragu et al. 2005; Lee and Elsayed 2005; Shah and Khanzode 2017; Zhang et al. 2017). Therefore, the sizing decision in a static problem is more strategic, and once the sizing decision is made, the warehouse space is not flexible to change. A shortage of the static problems is that these models do not capture the storage demand variation and uncertainty. To overcome this disadvantage of the static models, a few researchers focus on considering dynamic warehouse sizing problems. In the dynamic problems, the storage demands vary with time, and the decision-maker can change warehouse space over time by leasing additional space or closing a section of the warehouse. White and Francis (1971) may be the first work that considers the dynamic problem with probabilistic storage demand. Interestingly, they show that the dynamic problem can be formulated as a linear programming problem and transformed into a network flow problem. Then, Lowe, Francis, and Reinhardt (1979) propose an greedy algorithm contributing to solving the network flow problem efficiently. Rao and Rao (1998) consider a dynamic problem with concave cost function and show that the dynamic problem can be efficiently solved by a dynamic programming. In addition, some researchers study dynamic warehouse sizing problem based on queueing models. Huang et al. (2014) consider an integrated model for site selection and warehouse sizing in a two-stage network, and each warehouse is modelled as an $M/G/c$ queueing system. Gong et al. (2013) study the design of self-storage warehouse to maximise revenue subject to volatile storage demands. In their model, the renting process of each storage unit is presented by independent $M/G/x/x$ queueing loss system. Yuan et al. (2016) also use a queueing model to study the sizing of public storage warehouses. With the development of modern warehouses, a few scholars focus on studying the sizing problem in automated warehouses, for example, Inman (2003) considers the sizing problem of AS/RS system with automotive assembly sequences.

This paper also concentrates on addressing a dynamic warehouse sizing problem with uncertain storage demand, which originates from the uncertain product demand. Compared with the above mentioned studies, this paper contributes to proposing a dynamic model incorporating dynamically updated demand forecast and investigating how the demand forecast affects optimal sizing decisions. Choi (2009) also considers a warehouse sizing problem with demand forecast. However, the demands in that model are deterministic, and the demand forecast is static (fixed across the planning horizon) rather than dynamic updating, which are far from the reality. Unlike Choi (2009), this paper captures the demand uncertainty and dynamic evolvement of demand forecasts. To the best of our knowledge, this paper is probably the first attempt to address dynamic warehouse sizing problem with dynamically updating demand forecast. In addition to the demand forecast, the warehouse operator in this model can better handle the demand uncertainty via the flexibility of leasing contract. As one kind of operational flexibilities, contract flexibility has been demonstrated to be an effective way to handle operational uncertainty (Tachizawa and Gimenez 2009; Yao et al. 2010; Mardan et al. 2015; Ping and Liu 2015). Although flexible (leasing) contracts have been widely used in warehousing industry (Choi 2009), there are few papers on warehouse sizing problem with contract flexibility. This paper also contributes to establishing a dynamic warehouse sizing model with contract flexibility, which further allows us to investigate interaction between demand forecast and contract flexibility.

On the other hand, this paper is also related to the literature on dynamic inventory management with demand forecast (Barrow and Kourentzes 2016; Prak, Teunter, and Syntetos 2017). A central issue discussed by this stream of literature is that how does demand forecast affect optimal inventory decision. For example, Gallego and Özer (2001) and Özer and Wei (2004) consider the advance demand information in a multi-period inventory system without and with production capacity. They both show the monotonicity of the optimal inventory decision w.r.t. the advance demand information flow. Actually, the advance demand information is a special case of martingale model of forecast evolution (MMFE), which is widely used to characterise the demand forecast updating (Heath and Jackson 1994; Chen and Lee 2009; Sechan and Özer 2013). Wang, Atasu, and Kurtulus (2012) consider a newsvendor problem with demand forecast and multi-ordering opportunities. In addition, some researchers also use MMFE to characterise the supply forecast updating, e.g. Altug and Muharremoglu (2011).

Comparing with these studies, we contribute to characterising the monotonicity of optimal warehouse sizing decision w.r.t. the forecasted demand, which has not been captured by the existing studies. In addition, it is interesting to find that strategic choice of the nominal size does not necessarily increase with the forecasted demand, which contradicts to the monotonicity result established by the current literature.

3. Formulation

We consider a single product and periodic review inventory system with a finite horizon of N periods. Before the horizon starts, the system manager signs a contract with a TPL and announces a nominal size s (ft^2) of space to rent for storing her product. This nominal size serves as a demand forecast for the TPL. During the horizon, the manager determines the order quantity and warehouse size at the beginning of each period. In particular, the manager is allowed to adjust the size within a range $[(1-\beta)s, (1+\alpha)s]$, where exogenous parameters α and β measure the flexibility in storage capacity expansion (reduction). This kind of flexibility has been commonly used in practice and OM literature (Chen, Hum, and Sun 2001; Ben-Tal et al. 2005). In particular, when $\alpha = +\infty, \beta = 1$, the contract is fully flexible and allows the manager to rent the space that she exactly wants to use. In modern warehousing industry, a few TPLs (especially, self-storage providers) offer these fully flexible contracts (e.g. PLC Big Yellow Group 2015). On the other hand, when $\alpha = \beta = 0$, the contract degenerates into the classical contract without any flexibility. For notational convenience, we define $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.

After the horizon begins, the manager makes ordering decisions based on dynamically updating demand forecasts. The updating process is represented by a MMFE (Altug and Muharremoglu 2011; Wang, Atasu, and Kurtulus 2012). Specifically, at the beginning of period 1, the manager receives a sequence of estimated demands $\boldsymbol{d} = (d_1, \ldots, d_N)$ of the subsequent N periods, which serve as the initial demand forecast. The exact demand of period $n(= 1, \ldots, N)$ is given by

$$D_n = d_n + \sum_{i=1\vee(n-I)}^{n} \xi_{i,n},$$

where I is the length of information horizon, and $\xi_{i,n}$ is the adjustment for the demand of nth period observed at the end of period i. Before observation, $\xi_{i,n}$ is i.i.d. random variable with mean zero. Under these assumptions, the manager at the beginning of period n has the following vector of observed demands in hand:

$$\boldsymbol{d}_n = \{d_{n,n}, \ldots, d_{n,N}\},$$

where $d_{n,j}$ is the sum of observed demand signals, more specifically,

$$d_{n,j} = \begin{cases} d_j + \sum_{i=j-I}^{n-1} \xi_{i,j}, & j \in \{n, \ldots, n+I-1\}, \\ d_j, & j \in \{n+I, \ldots, N\}. \end{cases} \tag{1}$$

Thus, $D_j (j = n, \ldots, N)$ can be equivalently formulated as the sum of the observed part $d_{n,j}$ and unobserved part $U_{n,j}$, where

$$U_{n,j} = \begin{cases} \sum_{i=n}^{j} \xi_{i,j}, & j \in \{n, \ldots, n+I-1\}, \\ \sum_{i=j-I}^{j} \xi_{i,j}, & j \in \{n+I, \ldots, N\}. \end{cases} \tag{2}$$

At the beginning of period $n+1$, the manager further observes $\boldsymbol{\xi}_n = (\xi_{n,n}, \xi_{n,n+1}, \ldots, \xi_{n,n+I})$, and the vector of observed demands are updated as

$$\boldsymbol{D}_{n+1} = \begin{cases} (d_{n,n+1} + \xi_{n,n+1}, \ldots, \xi_{n,n+I}, \ldots, d_N), & n+I < N, \\ (d_{n,n+1} + \xi_{n,n+1}, \ldots, \xi_{n,n+I}, \ldots, d_{n,N} + \xi_{n,N}), & n+I \geq N. \end{cases} \tag{3}$$

We next specify the cost structure of this model. At the beginning of period n, the manager places an order of quantity q_n to replenish inventory according to on-hand inventory x_n and the demand forecast $\boldsymbol{d}_n$, and receives the ordered products immediately. The ordering cost is c per unit, and the size of the product is v per unit. Without loss of generality, c and v are normalised to be 0 and 1, respectively. The manager meanwhile determines the storage capacity $(1-\beta)s \leq z_n \leq (1+\alpha)s$ at rental cost r to store inventory $y_n = x_n + q_n$. If the contract space is not enough $(y_n > z_n)$, the overflow space will be used, which leads to the overflow cost $o(> r)$. During period n, the demands D_n arrive and unsatisfied demands are fully backlogged. Define h and p as the holding and backlogging cost per unit. Then, the holding and backlogging cost function of period n is

$$\omega(y_n, d_{n,n} + \xi_{n,n}) = h(y_n - d_{n,n} - \xi_{n,n})^+ + p(d_{n,n} + \xi_{n,n} - y_n)^+. \tag{4}$$

Here, we provide an interpretation of the holding cost h. In our problem, since we treat the warehouse rent and inventory replenishment in a separate way, the holding cost does not include the rental here. Similar assumptions and justifications can also be found from Cormier and Gunn (1996a) and Mark, Jihong, and Chung-Piaw (2001). The manager aims to minimise the total discounted expected cost resulting from holding, backlogging and renting activities by making twofold decisions: (i) determining the order quantity q_n and warehouse size z_n at the beginning of each period (operational level), and (ii) determining the nominal size s before the horizon begins (strategic level).

Based on the above description, we are ready to formulate this problem. For the operational problem, we use $(x_n, s, \boldsymbol{d}_n)$ to track the system states at the beginning of period n. The single period cost function is

$$f(y_n, z_n) = \omega(y_n, d_{n,n} + \xi_{n,n}) + o\,(y_n - z_n)^+ + r z_n.$$

Then, the optimal cost-to-go function $V_n(x_n, s, \boldsymbol{d}_n)$ is given by

$$V_n(x_n, s, \boldsymbol{d}_n) = \min_{y_n, z_n} \mathrm{E}\left[f(y_n, z_n) + \gamma V_{n+1}\left(y_n - d_{n,n} - \xi_{n,n}, s, \boldsymbol{D}_{n+1}\right)\right], \tag{5}$$

$$\text{s.t.} \quad x_n \leq y_n,\ (1-\beta)s \leq z_n \leq (1+\alpha)s, \tag{6}$$

with $V_{N+1}(x_{N+1}, s, \boldsymbol{d}_{N+1}) \equiv 0$. Here, γ is the discount factor over periods. For the strategic problem, the optimal nominal size is given by solving

$$\min_{s \geq 0} \left[V_1(x_1, s, \boldsymbol{d}) - c(s)\right], \tag{7}$$

where $c(s)$ is a convex cost function of reserving the nominal space.

Throughout this paper, words 'increasing' and 'decreasing' are used in a weaker sense, that is, 'increasing' means 'nondecreasing', and 'decreasing' means 'nonincreasing'.

4. Preliminaries

Before presenting main analytical results, we introduce a property called $L^{\natural}$-convexity.

Definition 1 A M-dimension function $f : \mathscr{R}^M \to \mathscr{R}$ is called $L^\natural$-convex if the function $\phi(\boldsymbol{x}, \xi) = f(\boldsymbol{x} - \xi \boldsymbol{e})$ is submodular on $\mathscr{R}^M \times \mathscr{R}$.

Similar to submodularity, $L^\natural$-convexity is an important notion which has been widely used to characterise structural properties of economic systems. Note that $L^\natural$-convexity is a stronger notion than submodularity because it implies not only the submodularity but also the convexity and the diagonal dominance of the Hessian matrix of f in addition to submodularity; interested readers are referred to Zipkin (2008) and Chen, Pang, and Pan (2014) for detailed discussions.

In this paper, $L^\natural$-convexity is used to provide economic interpretation for some elements (like, inventory and demand forecast) in our warehouse systems and characterise some monotonicity of the optimal decisions w.r.t. these elements. The following results summarise a few useful properties associated with $L^\natural$-convexity:

LEMMA 1 *[Lemmas 1–3 of Zipkin (2008)] The following statements are true.*

(i) *If $f(\boldsymbol{x})$ is $L^\natural$-convex, so is $g(\boldsymbol{x}, \epsilon) = f(\boldsymbol{x} - \epsilon \boldsymbol{e})$.*
(ii) *If $f(\boldsymbol{x}, y)$ is $L^\natural$-convex, so is $g(\boldsymbol{x}) = \min_y f(\boldsymbol{x}, y)$.*
(iii) *If $f(\boldsymbol{x}, y)$ is $L^\natural$-convex, the largest optimizer $y(\boldsymbol{x}) = \arg\min_y f(\boldsymbol{x}, y)$ is increasing in $\boldsymbol{x}$, and $y(\boldsymbol{x} + \epsilon \boldsymbol{e}) \leq y(\boldsymbol{x}) + \epsilon$ for any $\epsilon > 0$.*

Here, we omit the proof of Lemma 1; interested readers are referred to Zipkin (2008) for a complete proof. In Lemma 1, Parts (i) and (ii) show two operations that preserve $L^\natural$-convexity. Particularly, $L^\natural$-convexity is preserved under minimisation. Part (iii) shows that the minimiser of a $L^\natural$-convex function is increasing in the parameters $\boldsymbol{x}$ and the bounded sensitivity. This result helps us characterise the monotonicity of renting and ordering decisions w.r.t. the contract flexibility parameters; see Proposition 4.

5. Operational and strategic decision analysis

In the section, the analysis generally follows a backwards induction, that is, we start with analysing the operational decisions during the horizon for any given the nominal size, and then concentrate on the strategic choice of the nominal size.

5.1 *Operational decisions*

The following result is a brief characterisation of the structure of the optimal cost-to-go function:

LEMMA 2 *For any $n = 1, \ldots, N$,*

(i) *$V_n(x_n, s)$ is increasing in x_n;*
(ii) *$V_n(x_n, s)$ is jointly convex in (x_n, s).*

Lemma 2 characterises the convexity of $V_n()$, which implies that optimal control policy during the horizon is a state-dependent and base-stock one. We next characterise the structure of the optimal policy.

Suppose that the manager has sufficient contract space for any inventory level after replenishment. In this case, the manager's optimal renting decision is given by $\max\{y, (1-\beta)s\}$ (the maximum of y and $(1-\beta)s$ is due to the lower bound $z \geq (1-\beta)s$). In other words, the manger only needs to concentrate on how much to order, and her objective function is simplified as

$$\pi_n^U(y) = \mathrm{E}\left[\omega(y, d_{n,n} + \xi_{n,n}) + ry + \gamma V_{n+1}\left(y - d_{n,n} - \xi_{n,n}, s, \boldsymbol{D}_{n+1}\right)\right]. \quad (8)$$

On the other hand, if the contract space is not enough and the manager has to use overflow warehouse, then the manager's optimal renting decision is $(1+\alpha)s$, and the overflow space to rent is $y - (1+\alpha)s$. In this case, the manager also focuses on ordering decision, while the objective function becomes

$$\pi_n^L(y) = \mathrm{E}\left[\omega(y, d_{n,n} + \xi_{n,n}) + oy + \gamma V_{n+1}\left(y - d_{n,n} - \xi_{n,n}, s, \boldsymbol{D}_{n+1}\right)\right]. \quad (9)$$

Note that the major difference between $\pi_n^U(y)$ and $\pi_n^L(y)$ is the warehousing cost parameters r and o. According to the convexity of $\pi_n^U(y)$ and $\pi_n^L(y)$, it is easy to show that $y_n^U = \arg\max_y \pi_n^U(y) > y_n^L = \arg\max_y \pi_n^L(y)$. Then, we are ready to summarise the optimal policy for the operational problem (5)–(6).

PROPOSITION 1 *For any $n = 1, \ldots, N$, define $\tilde{y}_n^U = \max\{y_n^U, x_n\}$ and $\tilde{y}_n^L = \max\{y_n^L, x_n\}$, and the optimal order quantity and warehouse size (q_n^*, z_n^*) are given by*

(i) *if $\tilde{y}_n^U \leq (1+\alpha)s$, $q_n^* = \tilde{y}_n^U - x_n$ and $z_n^* = \max\{(1-\beta)s, \tilde{y}_n^U\}$;*

(ii) *if* $\tilde{y}_n^L < (1+\alpha)s < \tilde{y}_n^U$, $q_n^* = (1+\alpha)s - x_n$ *and* $z_n^* = (1+\alpha)s$*;*
(iii) *if* $\tilde{y}_n^L \geq (1+\alpha)s$, $q_n^* = \tilde{y}_n^L - x_n$ *and* $z_n^* = (1+\alpha)s$.

Proposition 1 specifies the structure of optimal control policy according to the different relationships of $\left((1+\alpha)s, \tilde{y}_n^L, \tilde{y}_n^U\right)$. Specifically, Case (i) means that the manager has enough contract space for storing inventory $\tilde{y}_n^U$, where $\tilde{y}_n^U$ is the manager's optimal order-up-level without using overflow space. In this case, it is optimal for the manager to order $\tilde{y}_n^U - x_n$ and rent contract space $\max\{(1-\beta)s, \tilde{y}_n^U\}$. Case (iii) indicates that the maximum contract space is such small that the manager has to use overflow space for storing excess products. Then, the manager should order $\tilde{y}_n^L - x_n$ and fill up the maximum contract space. In addition, Case (ii) is interesting and means that the maximum contract space is not enough and it is not cost effective to use any overflow space (because the expansion of the total space by using overflow space results in more warehousing costs). In this case, it is optimal for the manager to order so many products to fill up the maximum contract space. We next analyse how the nominal space and the demand forecast affect the operational decisions.

PROPOSITION 2 *For any* $n = 1, \ldots, N$,

(i) $V_n(x_n, s)$ *is submodular;*
(ii) $y_n^*(s)$ *and* $z_n^*(s)$ *are increasing in* s.

Proposition 2 indicates that on-hand inventory and the nominal space are economic substitutes, which implies that optimal order quantity and warehouse size are both increasing with the nominal space. An intuitive explanation is that a larger size of the nominal space means that the manager has more storage capacity to store more products, which leads to a larger order quantity. The following result characterises the behaviour of optimal operational decisions w.r.t. the demand forecast.

PROPOSITION 3 *For any* $n = 1, \ldots, N$,

(i) $V_n(x_n, d_{n,j})$ *is submodular.*
(ii) $y_n^*(\boldsymbol{d}_n)$ *and* $z_n^*(\boldsymbol{d}_n)$ *are increasing in* $\boldsymbol{d}_n$.
(iii) $\epsilon \geq q_n^*(\boldsymbol{d}_n + \epsilon \boldsymbol{e}_1) - q_n^*(\boldsymbol{d}_n) \geq q_n^*(\boldsymbol{d}_n + \epsilon \boldsymbol{e}_2) - q_n^*(\boldsymbol{d}_n) \geq \ldots \geq q_n^*(\boldsymbol{d}_n + \epsilon \boldsymbol{e}_I) - q_n^*(\boldsymbol{d}_n) \geq 0$.
(iv) $\epsilon \geq z_n^*(\boldsymbol{d}_n + \epsilon \boldsymbol{e}_1) - z_n^*(\boldsymbol{d}_n) \geq z_n^*(\boldsymbol{d}_n + \epsilon \boldsymbol{e}_2) - z_n^*(\boldsymbol{d}_n) \geq \ldots \geq z_n^*(\boldsymbol{d}_n + \epsilon \boldsymbol{e}_I) - z_n^*(\boldsymbol{d}_n) \geq 0$, *where* $\boldsymbol{e}_i$ *is the is a vector with 1 in its* i*th component and zero in all the other components.*

Proposition 3 shows that both the order quantity and warehouse size are increasing with the forecasted demands, that is, the manager needs to order and rent more when the demands are forecasted to grow. Moreover, Parts (iii) and (iv) show the bounded sensitivity of optimal decisions, and the forecasted demand that is closer to the current period has stronger impacts on optimal decisions. This result is an extension of the monotonicity result established by Gallego and Özer (2001), Özer and Wei (2004).

COROLLARY 1 *Define* $y_n^M = \arg\max_y \mathrm{E}\left[\omega(y, d_{n,n} + \xi_{n,n})\right]$, *and if* $(1+\alpha)s \geq x_n \geq y_n^M$, *then* $q_n^* = 0$, $z_n^* = x_n$.

In Corollary 1, y_n^M is actually upper bound on y_n^U (see Part (i) of Lemma 2). As a result, when $(1+\alpha)s \geq x_n \geq y_n^M$, it is optimal for the manager to order nothing and rent warehouse space just for storing on-hand inventory. The significance of Corollary 1 is twofold. It presents a condition for simplifying the computation of optimal decisions. More importantly, it indicates that the value of the demand forecast depends on the nominal warehouse space. Under the condition presented by Corollary 1, only the forecasted demand of the current period d_n is necessary, and the forecasted demands $(d_{n,n+1}, \ldots, d_{n,n+I})$ can be ignored. In other words, the value of $(d_{n,n+1}, \ldots, d_{n,n+I})$ is zero for the decision-making at the current period.

We next analyse how the flexibility parameters α and β affect the manager's decision-making during the horizon. For technical convenience, we define $s_\alpha = (1+\alpha)s$, $s_\beta = (1-\beta)s$, and study the system under states (x_n, s_α, s_β). Then, we have

$$V_n(x_n, s_\alpha, s_\beta, \boldsymbol{d}_n) = \min_{y_n, z_n} \mathrm{E}\left[f(y_n, z_n) + \gamma V_{n+1}\left(y_n - d_{n,n} - \xi_{n,n}, s_\alpha, s_\beta, \boldsymbol{D}_{n+1}\right)\right],$$
$$\text{s.t.} \qquad x_n \leq y_n, s_\beta \leq z_n \leq s_\alpha.$$

PROPOSITION 4 *For any* $n = 1, \ldots, N$,

(i) $V_n(x_n, s_\alpha, s_\beta)$ *is decreasing in* s_α, *while increasing in* s_β*;*
(ii) $V_n(x_n, s_\alpha, s_\beta)$ *is* $L^\natural$*-convex in* (x_n, s_α, s_β)*;*
(iii) $y_n^*(\alpha, \beta)$ *and* $z_n^*(\alpha, \beta)$ *are both increasing (decreasing) with* α (β).

Part (iii) of Proposition 4 shows that a larger value of α encourages the manager to order and rent more because the manager has more flexibility in expanding the storage space. On the other hand, when the manager has more flexibility in

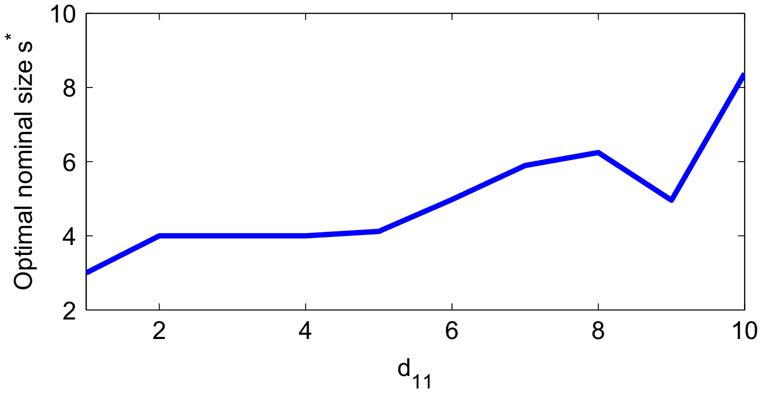

Figure 1. The behaviour of optimal nominal size s^* w.r.t. d_{11} in a two-period example with $\alpha = \beta = 0, h = 1, p = 2, o = 1.5, c = 1.5, \gamma = 0.95$ and demand $\xi \in \{0, 2\}$ with probabilities $\{0.4, 0.6\}$.

reducing the storage space, she will order and rent less. This result indicates that more contract flexibility will enhance the manager's flexibility in decision-making.

5.2 *Strategic decisions*

In this section, we address the manager's strategic choice of the nominal size s^* by solving Problem (7). Based on Lemma 1, it is easy to obtain the convexity of $V_1(x_1, s, \boldsymbol{d})$ w.r.t. s. Hence, s^* can be numerically solved by a local search algorithm, such as golden search.

We next focus on characterising the behaviour of $s^*(\boldsymbol{d})$ w.r.t. $\boldsymbol{d}$. An intuitive guess is that s^* is increasing in $\boldsymbol{d}$ based on the thought that more space is necessary to rent when the forecasted demands increase. The following result shows that the monotonicity does not necessarily hold:

PROPOSITION 5 *$s^*(\boldsymbol{d})$ is increasing in d_N, while not necessarily increasing in $(d_1, \ldots, d_{N-1})$.*

The first part of Proposition 5 partially characterises the monotonicity of $s^*(\boldsymbol{d})$ w.r.t. the forecasted demand of the last period. The proof is based on the supermodularity of $V_1(x_1, s, d_N)$. (This result is easy to prove by induction; therefore, we omit it here.) However, this monotonicity does not necessarily hold for $(d_1, \ldots, d_{N-1})$. Figure 1 depicts the behaviour of optimal nominal size s^* w.r.t. d_{11} in a two-period example, and $s^*(d_{11})$ sometimes decreases with d_{11}. The reason for this counter-intuitive result is that although the optimal order quantity $q_1^*(d_{11})$ will increase with d_{11}, the resulting inventory level $x_n + q_1^*(d_{11}) - d_{11} - \xi_{11}$ at the beginning of period 2 may decrease with d_{11}. As a result, it might be profitable for the manager to reduce the nominal size since the on-hand inventory level of next period would decrease with d_n $(n = 1, \ldots, N-1)$. In addition, the unsold inventory of period N does not affect the choice of optimal nominal size. Therefore, $s^*(d_N)$ is increasing with d_N.

We next analyse how the flexibility parameters α and β affect the choice of optimal nominal size.

PROPOSITION 6 (i) *$s^*(r)$ is decreasing in r.* (ii) *When $\alpha = \infty$, $V_1(s, \beta)$ is submodular, and $s^*(\beta)$ is increasing in β.*

Proposition 6 shows the monotonicity of optimal nominal size s^* w.r.t. the rental r and flexibility parameter β. In particular, Part (ii) characterises the complementary relationship between s and β when the flexibility in expanding storage space is unlimited. However, the monotonicity result does not generally hold; see Figure 2. For example, when $\alpha \in [0, 0.1]$, optimal nominal size actually decreases with α, which contradicts with the intuition that s^* decreases with α. This counter-intuitive result indicates that when α is relatively small ($\alpha \in [0, 0.1]$), obtaining a larger nominal size might benefit the manager by further enhancing the flexibility in expanding storage space.

6. Case studies: Values of Demand Forecast and contract flexibility

Via preceding analysis we find that both demand forecast and contract flexibility can help the manager better plan the warehouse sizing. In this section, we further address the Values of Demand Forecast and contract flexibility through a case

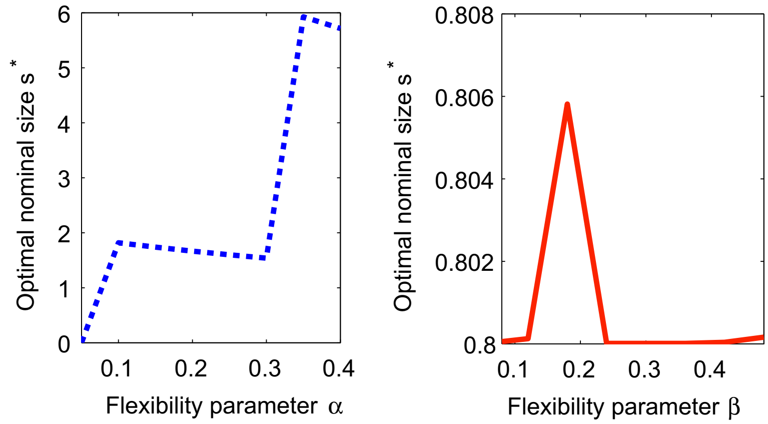

Figure 2. The non-monotonicity of optimal nominal size s^* w.r.t. the flexibility parameters α and β when $h = 1$, $p = 2$, $r = 0.5$, $\gamma = 0.95$ and $\xi \in \{10, 2, 8, 20, 15\}$ with probabilities $\{0.2, 0.2, 0.4, 0.1, 0.1\}$.

Table 1. Parameters.

Parameter	$h\ \left(\frac{\text{euros}}{\text{pallet}}\right)$	$p\ \left(\frac{\text{euros}}{\text{pallet}}\right)$	$r\ \left(\frac{\text{euros}}{\text{pallet}}\right)$	$\bar{c}\ \left(\frac{\text{euros}}{\text{pallet}}\right)$	$o\ \left(\frac{\text{euros}}{\text{pallet}}\right)$	γ
Value	1.9	12	6	2	10	0.95

Table 2. Optimal costs under different combinations of (α, β).

	β					
α	0.1	0.2	0.3	0.4	0.5	1
0.1	116,177	114,276	112,085	108,795	105,441	104,152
0.2	114,461	112,326	109,745	106,365	104,260	103,489
0.3	112,745	110,376	107,405	104,593	103,316	102,952
0.4	111,029	108,426	105,501	103,260	102,554	102,476
1	101,914	101,178	100,754	100,585	100,585	100,585
∞	101,610	100,874	100,450	100,385	100,385	100,385

study. In particular, we aim to address two questions: (i) How does demand forecast affect the VCF? (ii) How does contract flexibility affect the VDF?

Since this paper is based on the real-life problem studied by Choi (2009), we use the demand and cost data presented in Choi (2009) to initialise some key parameters in this case study. To be specific, an international manufacturer rents the warehouse space in a half-yearly contract, and determines the nominal space to rent based on a vector of estimated demands of subsequent six months $\boldsymbol{d} = (2200, 2200, 2100, 2200, 2000, 2000)$ (pallets). During the planning horizon, the manager receives a vector of forecasted demands $\boldsymbol{d}_n = (d_{n,n}, d_{n,n+1}, d_{n,n+2})$ at the beginning of period n (i.e. the length of the information horizon is $I = 3$). The demand adjustments $\{\xi_{i,j}\}$ are i.i.d. The reservation cost $c(s) = \bar{c}s$, where $\bar{c}$ is a positive constant. In particular, the demand and cost parameters are partially from the manufacturer's historical data and summarised in Table 1.

To measure the VCF, we consider a benchmark case where $\alpha = \beta = 0$ (that is, the nominal size is exactly the committed size), and compute the optimal cost V^B in the benchmark case. Then, we calculate the optimal costs $V(\alpha, \beta)$ under different combination of $\alpha \in \{0.1, 0.2, 0.3, 0.4, 1, +\infty\}$ and $\beta \in \{0.1, 0.2, 0.3, 0.4, 0.5, 1\}$, and difference between $V^B - V(\alpha, \beta)$ represents the VCF, which is upper bounded by $V^B - V(+\infty, 1)$. Table 2 lists the optimal costs under different combinations of (α, β). Clearly, a larger value of α or β (a higher flexibility) results in smaller inventory and warehousing cost. Combining the data listed in Table 2 and $V^B = 120,131$, we can compute and compare the VCFs in different scenarios, which is shown by Figure 3. Based on Figure 3 we can observe the diminishing return of enhancing contract flexibility. Moreover, it is interesting to find that the performance of the system when $\alpha = 1$ is very close to that when $\alpha = +\infty$, which implies that the manager may not necessarily obtain great flexibility in capacity expansion.

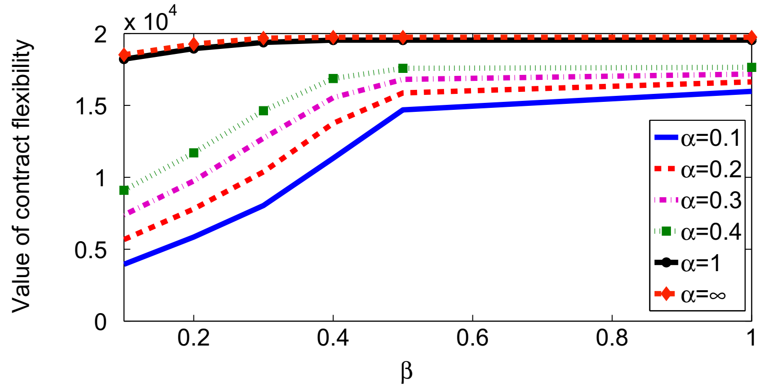

Figure 3. VCF of different combinations of (α, β).

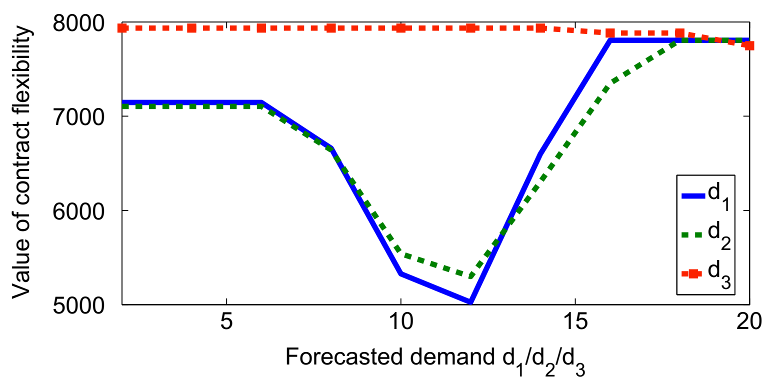

Figure 4. VCF when forecasted demands differ.

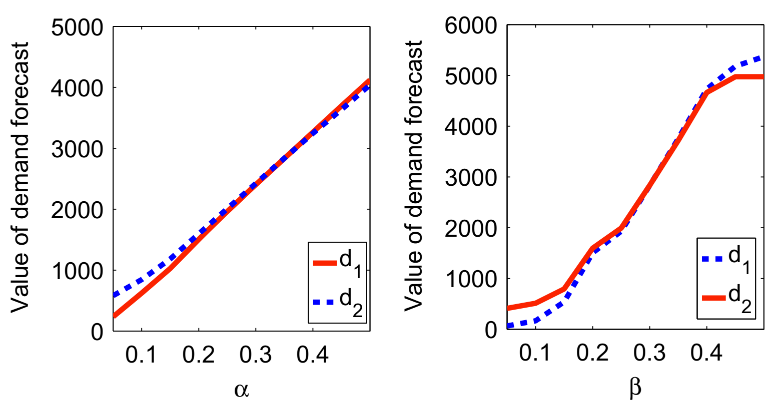

Figure 5. VCF under different combinations of (α, β).

Figure 4 shows how the initial demand forecast $\boldsymbol{d}_1$ affects the VCF. Interestingly, a larger value of the forecasted demand does not necessarily imply a higher VCF. For example, when d_1 changes from 1 to 20, the VCF decreases over [1,12] and increases over (13,20]. A possible explanation for this interesting observation is as follows. When d_1 is relatively small ($d_1 \in [1, 12]$), the demanded warehouse space is also relatively small. Then, and the flexibility of reducing space is more valuable for the manager. However, as d_1 increases, the demanded warehouse space increases, and the benefit resulting from reducing space becomes less significant. As d_1 continues to increase to a large value ($d_1 \in (12, 20]$), the flexibility of expanding space is more valuable for the manager. In this case, the benefit resulting from expanding space becomes more significant.

To evaluate the VDF, we use a benchmark in which the exact value of $\boldsymbol{d}_1$ is absent. For example, when evaluating the value of forecasted demand d_1, we use a random variable ξ that satisfies $\mathrm{E}[\xi] = d_1$, and compute the optimal value of $V^{BI}(\xi) = \max_s \mathrm{E}[V_1(x_1, \xi)]$. Then, the value of forecasted demand d_1 is represented by $V^{BI}(\xi) - \max_s V_1(x_1, d_1)$.

Figure 5 shows the values of forecasted demands d_1 and d_2 under different combinations (α, β). Clearly, the VDF increases with the contract flexility. This result implies that the VDF can be limited by low-level contract flexibility, or we can say that high-level contract flexibility helps the manager make full use of the demand forecast.

7. Conclusions

In this paper, we study a dynamic warehouse sizing problem with demand forecast and contract flexibility. In this problem, a manager needs to make both strategic and operational decisions. On the strategic level, the manager announces the nominal size of warehouse space to rent before the planning horizon begins when signing a contract with a TPL. On the operational level, the manager determines the order quantity and actual warehouse size at the beginning of each period during the horizon. In particular, the manager can adjust the warehouse space within a range according to updated demand forecast, which reflects the contract flexibility. In addition to the nominal size, the strength of the flexibility is measured by two exogenous parameters that characterize flexibility in capacity expansion and reduction.

By studying such a model, we investigate how the demand forecast and contract flexibility affect the strategic and operational decision-making. For any given nominal size, we show the monotonicity of the operational decisions w.r.t. the forecasted demand and contract flexibility. These results emphasise the significance of demand forecast and contract flexibility on optimising operational decisions. However, these monotonicity results do not necessarily hold for the strategic decision. For example, it is interesting to find that the optimal nominal size may decrease with the forecasted demands, which contradicts to intuition that more space should be rented when the space demand is forecasted to increase.

Although all the analytical results in this paper are derived based on the assumption that unsatisfied demands are assumed to be backlogged, these results preserve to be true in a lost-sale model. In a lost-sale model, the transition of the inventory state is modified as $x_{n+1} = (x_n - d_{n,n} - \xi_{n,n})^+$. Noticing that the optimal cost-to-go function is increasing in the on-hand inventory level, it is easy to prove the convexity and submodularity of the optimal cost-to-go function by following classical techniques in analysing lost-sale inventory system. In addition, these results remain valid under a more general convex overflow cost function.

We also propose two potential extensions for future research. One is to endogenise the flexibility parameter as a strategic decision before the planning horizon begins. In this case, we can concentrate on analysing how demand forecast affects the choice of optimal flexibility level. The other is to explicitly consider delivery leadtime. Note that the classical formulation of dynamic backlogging inventory system with leadtime (by using the concept of inventory position) is not valid in this case. Therefore, it is necessary to track the outstanding orders, and the state dimension could be very large. Thus, it may be necessary to develop efficient heuristics (via state reduction, or approximate dynamic programming) for solve the warehouse sizing problem with leadtime.

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Appendix 1

Proof of Lemma 2 By induction. When $n = N+1$, the results trivially hold. Inductively assume that (i) and (ii) hold for $V_{n+1}()$, and then it is straightforward to find that (i) also holds for $V_n()$. To prove the convexity of $V_n()$, it suffices to show that for any $y_1, y_2, s_1, s_2, \lambda_1, \lambda_2$ $(\lambda_1 + \lambda_2 = 1)$,

$$\begin{aligned}\lambda_1 \mathrm{E}\left[V_{n+1}\left(y_1 - d_{n,n} - \xi_{n,n}, s_1\right)\right] + \lambda_2 \mathrm{E}\left[V_{n+1}\left(y_2 - d_{n,n} - \xi_{n,n}, s_2\right)\right] \\ \geq \mathrm{E}\left[V_{n+1}\left(\lambda_1 y_1 + \lambda_2 y_2 - d_{n,n} - \xi_{n,n}, \lambda_1 s_1 + \lambda_2 s_2\right)\right],\end{aligned}$$

where the inequality results from the joint convexity of $V_{n+1}()$. Then, since the constraints $x_n \leq y_n$, $(1-\beta)s \leq z_n \leq (1+\alpha)s$ are convex, it is easy to show the joint convexity of $V_n()$ by applying Proposition B4 of Heyman and Sobel (2003).

Proof of Proposition 1 We prove the result based on the convexity of functions (8) and (9). Note that (8) is the manager's objective function when she has sufficient contract space for inventory storage, and $\tilde{y}_n^U = \max\{y_n^U, x_n\}$ is her optimal order-up-to inventory level in this case. However, if the contract space is not enough and the manager has to use the overflow space, her objective function becomes (9), and her optimal order-up-to inventory level is $\tilde{y}_n^L = \max\{y_n^L, x_n\}$ in this case.

If $\tilde{y}_n^U \leq (1+\alpha)s$, which means the manager can reach $\tilde{y}_n^U$ without using overflow warehouse, then it is definitely optimal for the manager to order $q_n^* = \tilde{y}_n^U - x_n$ and use contract space $z_n^* = \max\{(1-\beta)s, \tilde{y}_n^U\}$. (Here, the maximum of $(1-\beta)s$ and $\tilde{y}_n^U$ results from the lower bound constraint $z \geq (1-\beta)s$.) If $\tilde{y}_n^L \geq (1+\alpha)s$, which means that the manager has to use overflow warehouse for storing excess products, then $q_n^* = \tilde{y}_n^L - x_n$ and $z_n^* = (1+\alpha)s$. However, if $\tilde{y}_n^L \leq (1+\alpha)s \leq \tilde{y}_n^U$, which means that the contract space is not enough for reaching $\tilde{y}_n^U$, and the overflow space is not cost effective to use (because the expansion of the total space by using overflow space leads to more warehousing costs), then it is optimal for the manager to use up the contract space and order as many products as possible to fill up the maximum contract space, i.e. $q_n^* = (1+\alpha)s - x_n$ and $z_n^* = (1+\alpha)s$.

Proof of Proposition 2 By induction. When $n = N+1$, the result trivially holds. Inductively assume that $V_{n+1}()$ is submodular. Then, it is easy to show that function

$$f(y, z, s) = \mathrm{E}\left[\omega(y, d_{n,n} + \xi_{n,n}) + o\,(y-z)^+ + rz + \gamma V_{n+1}\left(y - d_{n,n} - \xi_{n,n}, s\right)\right],$$

is submodular in (y, z, s). Because the constraint set is a lattice, we have $V_n()$ is also submodular by applying Theorem 3.10 of Topkis (1998). Part (ii) is directly implied by the submodularity of $f(y, z, s)$.

Proof of Proposition 3 To prove the result, we define $\bar{\boldsymbol{d}}_n = -\boldsymbol{d}_n$, $\bar{q}_n = q_n - z_n$, and let $\bar{V}_n(x_n, \bar{d}_n)$ be optimal cost function under states $(x_n, \bar{d}_n)$. Then, we have

$$\begin{aligned}\bar{V}_n(x_n, \bar{\boldsymbol{d}}_n) = \min_{\bar{q}_n, z_n} \mathrm{E}\big[&\omega(z_n + \bar{q}_n + x_n, -\bar{d}_{n,n} - \xi_{n,n}) + o\,(\bar{q}_n + x_n)^+ - r z_n \\ &+ \gamma \bar{V}_{n+1}\left(z_n + \bar{q}_n + x_n + \bar{d}_{n,n} - \xi_{n,n}, \bar{\boldsymbol{D}}_{n+1}\right)\big], \\ \text{s.t.} \quad & z_n + \bar{q}_n \geq 0, (1+\alpha)s \geq z_n \geq (1-\beta)s\end{aligned}$$

Then, it is easy to prove that $\bar{V}_n(x_n, \bar{\boldsymbol{d}}_n)$ is multimodular in $(x_n, \bar{\boldsymbol{d}}_n)$, which implies Part (i). (For the definition of multimodular and its application, readers are referred to Li and Peiwen (2014) for an excellent introduction.) Moreover, according to Theorem 1 of Li and Peiwen (2014), we have both $\bar{q}_n^*\left(\bar{\boldsymbol{d}}_n\right)$ and $z_n^*\left(\bar{\boldsymbol{d}}_n\right)$ are decreasing in $\bar{\boldsymbol{d}}_n$, which further indicates Parts (ii), (iii), and (iv).

Proof of Proposition 4 By induction. When $n = N+1$, the results trivially hold. Inductively assume the results hold for $V_{n+1}()$. Based on the argument that the constraint set $\{z_n | s_\beta \leq z_n \leq s_\alpha\}$ becomes larger (smaller) as s_α (s_β) increases, it is straightforward to obtain Part (i). For Part (ii), it suffices to show that the submodularity of

$$V_n(x_n-\epsilon, s_\alpha-\epsilon, s_\beta-\epsilon) = \min_{y_n,z_n} \mathrm{E}\Big[\omega(y_n^\epsilon-\epsilon, D_n) + o\left(y_n^\epsilon - z_n^\epsilon\right)^+ + r(z_n^\epsilon-\epsilon) + \gamma V_{n+1}\left(y_n^\epsilon-\epsilon-D_n, s_\alpha-\epsilon, s_\beta-\epsilon\right)\Big],$$

$$\text{s.t.} \qquad x_n \le y_n^\epsilon, s_\beta \le z_n^\epsilon \le s_\alpha,$$

where $y_n^\epsilon = y_n + \epsilon, z_n^\epsilon = z_n + \epsilon$; see Definition 1. Based on the $L^\natural$-convexity of $V_{n+1}()$, it is easy to prove the $L^\natural$-convexity of $\hat{V}_n()$ by the preservation property under minimisation (see Lemma 1) , and both y_n^* and z_n^* are increasing with s_α and s_β, which implies that $y_n^*(\alpha,\beta)$ and $z_n^*(\alpha,\beta)$ are increasing (decreasing) with α (β).

Proof of Proposition 5 To prove (i), it suffices to show that $\bar{V}_n(x_n, s, \bar{r})$ is submodular, where $\bar{r} = -r$. Here, we prove the result by induction. When $n = N + 1$, the result trivially holds. Inductively assume the result holds for $\bar{V}_{n+1}()$. Then, we have

$$\bar{V}_n(x_n, s, \bar{r}) = \min_{y_n,z_n} \mathrm{E}\left[\omega(y_n, d_{n,n}+\xi_{n,n}) + o\,(y_n - z_n)^+ - \bar{r} z_n + \gamma V_{n+1}\left(y_n - d_{n,n} - \xi_{n,n}, s, \bar{r}\right)\right],$$

$$\text{s.t.} \qquad x_n \le y_n, (1-\beta)s \le z_n \le (1+\alpha)s.$$

Then, following the proof of Proposition 2, it is easy to find that $\bar{V}_n(x_n, s, \bar{r})$ is submodular.

When $\alpha = +\infty$, the problem of choosing optimal the nominal size becomes $\min_s[V_1(x_1, s, \beta)+c(s)] = \min_s[\hat{V}_1(x_1, s(1-\beta))+c(s)]$. It is easy to find that function $\hat{V}_1(x_1, s(1-\beta)) + c(s)$ is submodular i (β, s), which implies that $s^*(\beta)$ is increasing in β.